Critical set of eigenfunctions of the Laplacian

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Abstract

We give an upper bound for the $(n-1)$-dimensional Hausdorff measure of the critical set of eigenfunctions of the Laplacian on compact analytic Riemannian manifolds. This is the analog of H. Donnely and C. Fefferman [6] result on nodal set of eigenfunctions.

1 Introduction and statement of the results

Let (M, g) be a smooth, compact and connected, n-dimensional Riemannian manifold $(n \geq 2)$. For $u \in \mathcal{C}^1(M)$, we set

$$\mathcal{N}_u = \{x \in M : u(x) = 0\}$$

and

$$\mathcal{C}_u = \{x \in M : \nabla u(x) = 0\},$$

the nodal set of u and the critical set respectively. It is well kown that if u is a non trivial solution of second order linear elliptic equation then all zeros of u are of finite order ([1],[10]), and one can prove that the Hausdorff dimension of the nodal set $\mathcal{N}_u$ is at most $n-1$ (for example, see [4] or [8] for more precise results). When dealing with the eigenfunctions of the Laplacian :

$$-\Delta u = \lambda u, \tag{1.1}$$

S. T. Yau [15] has conjectured that

$$C_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}_u) \leq C_2\sqrt{\lambda}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure and C_1, C_2 are positives constants depending only upon M. In case that both the manifold and the metric are real analytic, the problem was solved by H. Donnelly and C. Fefferman [6], [7]. For smooth metric the only known upper bound result $(n \geq 3)$ is due to R. Hardt and L. Simon [8]. They proved that

$$\mathcal{H}^{n-1}(\mathcal{N}_u) \leq (c\sqrt{\lambda})^{c\sqrt{\lambda}}.$$

However this result doesn't seems to be optimal. Recently, different authors ([12] [5], [13]) obtained some lower bound with polynomial decrease in λ.

The critical set of eigenfunctions on the other hand is not so well understood (one could look at [16] for a quick survey). Generically eigenfunctions are Morse functions ([14]) and therefore the critical set consits in isolated points. Moreover, D. Jakobson and N. Nadirashvili [11] have shown that there exists in dimension two a sequence of eigenfunctions for which the number of critical points is uniformly bounded. However there exists simple examples for which the critical set has Hausdorff dimension $n-1$:

Example 1.1. Let (N, g) be a $(n-1)-$dimensional manifold and define $M = \mathbb{T}^1 \times N$ where $\mathbb{T}^1$ is the $1-$dimensionnal Torus with standard metric, and M is equipped with the product metric. The function $f_k(x, y) = \sin(2\pi kx)$ is an eigenfunction of Δ_M with eigenvalue $\lambda := k^2$. The critical set, $\mathcal{C}_{f_k}$, of f_k is therefore a set of dimension $n-1$. One should also note that $\mathcal{H}^{n-1}(\mathcal{C}_{f_k}) \geq C\sqrt{\lambda}$, where C depends only on M.

It is also easy to find some surface of revolution with critical set of dimension $(n-1)$, see [16] p 35. In the case of a critical set of dimension $n-1$ it seems interresting to obtained some upper bound on the $(n-1)-$dimensionnal Hausdorff measure. This is the goal of this paper. We will show that :

Theorem 1.2. *Let M be a n-dimensionnal, real analytic, compact, connected manifold with analytic metric. There exist $C > 0$ depending only on M such that for any non-constant solution u to* (1.1) *one has*

$$\mathcal{H}^{n-1}(\mathcal{C}_u) \leq C\sqrt{\lambda},$$

where $\mathcal{C}_u$ is the critical set of u.

The main ingredient in the proof of our theorem is the following doubling inequality on gradient of eigenfunctions

$$\|\nabla u\|_{B_{2r}} \leq e^{C\sqrt{\lambda}} \|\nabla u\|_{B_r}. \tag{1.2}$$

This estimate is a consequence of a general Carleman-type inequality which we also use to study the vanishing order of solutions to the Schrödinger equation in a related paper [2].

The paper is organised as follows. In the section 2 we deduce from [2] a Carleman estimate for the operator $\Delta + \lambda$ acting diagonally on vector valued functions. Using the compactness of M, this will allows us to derive in section 3 doubling estimates (1.2) using standard method of quantitative uniqueness. In section 3 we use the method developped by H. Donnelly and

C. Fefferman to show our estimate on the measure of the critical set in the case that M is an analytic manifold. One should note that the framework of this paper follows closely [2] until section 3, with some obvious adaptations to the vectorial case.

2 Carleman estimates

First we give a Carleman estimate on the scalar operator $\Delta + W$ with W of class $\mathcal{C}^1$, this can also be find in [2] and is write down here only for completness (and because of the electronic nature of this document).
Fix x_0 in M, and let : $r = r(x) = d(x, x_0)$ the Riemannian distance from x_0. We denote by $B_r(x_0)$ the geodesic ball centered at x_0 of radius r. We will denote by $\|\cdot\|$ the L^2 norm. Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\phi}$, where the function ϕ satisfy some convexity properties. Let us now define the weight function we will use.
For a fixed number ε such that $0 < \varepsilon < 1$ and $T_0 < 0$, we define the function f on $]-\infty, T_0[$ by $f(t) = t - e^{\varepsilon t}$. One can check easily that, for $|T_0|$ great enough, the function f verifies the following properties:

$$\begin{array}{l} 1 - \varepsilon e^{\varepsilon T_0} \le f'(t) \le 1 \qquad \forall t \in]-\infty, T_0[, \\ \lim\limits_{t \to -\infty} -e^{-t} f''(t) = +\infty. \end{array} \tag{2.1}$$

Finally we define $\phi(x) = -f(\ln r(x))$. Now we can state the main result of this section:

Theorem 2.1. *There exist positive constants R_0, C, C_1, C_2, which depend only on M, such that, for any $W \in \mathcal{C}^1(M)$, $x_0 \in M$, $u \in C_0^\infty(B_{R_0}(x_0) \setminus \{0\})$ and $\tau \ge C_1\sqrt{\|W\|_{\mathcal{C}^1}} + C_2$, one has*

$$C \left\| r^2 e^{\tau\phi} (\Delta u + W u) \right\| \ge \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \tag{2.2}$$

Moreover, if

$$\operatorname{supp}(u) \subset \{x \in M; r(x) \ge \delta > 0\},$$

then

$$\begin{array}{rcl} C \left\| r^2 e^{\tau\phi} (\Delta u + W u) \right\| & \ge & \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| \\ + \tau\delta \left\| r^{-1} e^{\tau\phi} u \right\| & + & \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \end{array} \tag{2.3}$$

Proof. Hereafter C, C_1, C_2 and c denote positive constants depending only upon M, though their values may change from one line to another. Without loss of generality, we may suppose that all functions are real. We now introduce the polar geodesic coordinates (r, θ) near x_0. Using Einstein notation, the Laplace operator takes the form :

$$r^2 \Delta u = r^2 \partial_r^2 u + r^2 \left(\partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u),$$

where $\partial_i = \dfrac{\partial}{\partial\theta_i}$ and for each fixed r, $\gamma_{ij}(r,\theta)$ is a metric on $\mathbb{S}^{n-1}$ and $\gamma = \det(\gamma_{ij})$.
Since (M,g) is smooth, we have for r small enough :

$$\begin{array}{rcl} \partial_r(\gamma^{ij}) & \leq & C(\gamma^{ij}) \quad \text{(in the sense of tensors);} \\ |\partial_r(\gamma)| & \leq & C; \\ C^{-1} \leq \gamma & \leq & C. \end{array} \tag{2.4}$$

Set $r = e^t$, we have $\dfrac{\partial}{\partial r} = e^{-t}\dfrac{\partial}{\partial t}$. Then the function u is supported in $]-\infty, T_0[\times\mathbb{S}^{n-1}$, where $|T_0|$ will be chosen large enough. In this new variables, we can write :

$$e^{2t}\Delta u = \partial_t^2 u + (n-2+\partial_t \ln\sqrt{\gamma})\partial_t u + \frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j u).$$

The conditions (2.4) become

$$\begin{array}{rcl} \partial_t(\gamma^{ij}) & \leq & Ce^t(\gamma^{ij}) \quad \text{(in the sense of tensors);} \\ |\partial_t(\gamma)| & \leq & Ce^t; \\ C^{-1} \leq \gamma & \leq & C. \end{array} \tag{2.5}$$

Now we introduce the conjugate operator :

$$\begin{array}{rcl} L_\tau(u) & = & e^{2t}e^{\tau\phi}\Delta(e^{-\tau\phi}u) + e^{2t}Wu \\ & = & \partial_t^2 u + \left(2\tau f' + n - 2 + \partial_t \ln\sqrt{\gamma}\right)\partial_t u \\ & + & \left(\tau^2 f'^2 + \tau f'' + (n-2)\tau f' + \tau\partial_t \ln\sqrt{\gamma} f'\right) u \\ & + & \Delta_\theta u + e^{2t}Wu, \end{array} \tag{2.6}$$

with

$$\Delta_\theta u = \frac{1}{\sqrt{\gamma}}\partial_i\left(\sqrt{\gamma}\gamma^{ij}\partial_j u\right).$$

It will be useful for us to introduce the following L^2 norm on $]-\infty, T_0[\times\mathbb{S}^{n-1}$:

$$\|V\|_f^2 = \int_{]-\infty,T_0[\times\mathbb{S}^{n-1}} V^2\sqrt{\gamma} f'^{-3} dt d\theta,$$

where $d\theta$ is the usual measure on $\mathbb{S}^{n-1}$. The corresponding inner product is denoted by $\langle\cdot,\cdot\rangle_f$, *i.e*

$$\langle u, v\rangle_f = \int uv\sqrt{\gamma} f'^{-3} dt d\theta.$$

We will estimate from below $\|L_\tau u\|_f^2$ by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of

τ and exponenial decay when t goes to $-\infty$. First by triangular inequality one has

$$\|L_\tau(u)\|_f \geq I - II, \tag{2.7}$$

with

$$\begin{array}{rcl} I &=& \left\|\partial_t^2 u + 2\tau f'\partial_t u + \tau^2 {f'}^2 u + e^{2t}Wu + \Delta_\theta u\right\|_f, \\ II &=& \left\|\tau f'' u + (n-2)\tau f' u + \tau\partial_t \ln\sqrt{\gamma} f' u\right\|_f \\ &+& \left\|(n-2)\partial_t u + \partial_t \ln\sqrt{\gamma}\partial_t u\right\|_f. \end{array} \tag{2.8}$$

We will be able to absorb II later. Then we compute I^2 :

$$I^2 = I_1 + I_2 + I_3,$$

with

$$\begin{array}{rcl} I_1 &=& \|\partial_t^2 u + (\tau^2 {f'}^2 + e^{2t}W)u + \Delta_\theta u\|_f^2 \\ I_2 &=& \|2\tau f'\partial_t u\|_f^2 \\ I_3 &=& 2\left\langle 2\tau f'\partial_t u, \partial_t^2 u + \tau^2 {f'}^2 u + e^{2t}Wu + \Delta_\theta u\right\rangle_f \end{array} \tag{2.9}$$

In order to compute I_3 we write it in a convenient way:

$$I_3 = J_1 + J_2 + J_3, \tag{2.10}$$

where the integrals J_i are defined by :

$$\begin{array}{rcl} J_1 &=& 2\tau \int f'\partial_t(|\partial_t u|^2){f'}^{-3}\sqrt{\gamma}dtd\theta \\ J_2 &=& 4\tau \int f'\partial_t u\partial_i\left(\sqrt{\gamma}\gamma^{ij}\partial_j u\right){f'}^{-3}dtd\theta \\ J_3 &=& \int\left(2\tau^3(f')^3 + 2\tau f' e^{2t}W\right)2u\partial_t u {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{array} \tag{2.11}$$

Now we will use integration by parts to estimate each terms of (2.11). Note that f is radial and that $2\partial_t u\partial_t^2 u = \partial_t(|\partial_t u|^2)$. We find that :

$$\begin{array}{rcl} J_1 &=& \int(4\tau f'')|\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ &-& \int 2\tau f'\partial_t \ln\sqrt{\gamma}|\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{array}$$

The conditions (2.5) imply that $|\partial_t \ln\sqrt{\gamma}| \leq Ce^t$. Then properties (2.1) on f gives, for large $|T_0|$ that $|\partial_t \ln\sqrt{\gamma}|$ is small compared to $|f''|$. Then one has

$$J_1 \geq -c\tau \int |f''| \cdot |\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \tag{2.12}$$

Now in order to estimate J_2 we first integrate by parts with respect to ∂_i :

$$J_2 = -2\int 2\tau f'\partial_t\partial_i u\gamma^{ij}\partial_j u {f'}^{-3}\sqrt{\gamma}dtd\theta.$$

Then we integrate by parts with respect to ∂_t. We get :

$$\begin{array}{rcl} J_2 &=& -4\tau \int f''\gamma^{ij}\partial_i u\partial_j u {f'}^{-3}\sqrt{\gamma}dtd\theta \\ &+& \int 2\tau f'\partial_t \ln\sqrt{\gamma}\gamma^{ij}\partial_i u\partial_j u {f'}^{-3}\sqrt{\gamma}dtd\theta \\ &+& \int 2\tau f'\partial_t(\gamma^{ij})\partial_i u\partial_j u {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{array}$$

We denote $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$. Now using that $-f''$ is non-negative and τ is large, the conditions (2.1) and (2.5) gives for $|T_0|$ large enough:

$$J_2 \geq 3\tau \int |f''| \cdot |D_\theta u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta. \tag{2.13}$$

Similarly computation of J_3 gives :

$$\begin{array}{rcl} J_3 & = & -2\int \tau^3 \partial_t \ln(\sqrt{\gamma}) u^2 \sqrt{\gamma} dt d\theta \\ & - & \int (4f' - 4f'' + 2f' \partial_t \ln \sqrt{\gamma}) \tau e^{2t} W u^2 {f'}^{-3} \sqrt{\gamma} dt d\theta \\ & - & \int 2\tau f' e^{2t} \partial_t W |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta. \end{array} \tag{2.14}$$

Now we assume that

$$\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2. \tag{2.15}$$

From (2.1) and (2.5) one can see that if C_1, C_2 and $|T_0|$ are large enough, then

$$J_3 \geq -c\tau^3 \int e^t |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta. \tag{2.16}$$

Thus far, using (2.12),(2.13) and (2.16), we have :

$$\begin{array}{rcl} I_3 & \geq & 3\tau \int |f''| \, |D_\theta u|^2 \, {f'}^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 {f'}^{-3} \sqrt{\gamma} dt d\theta \\ & - & c\tau \int |f''| \, |\partial_t u|^2 \, {f'}^{-3} \sqrt{\gamma} dt d\theta. \end{array} \tag{2.17}$$

Now we consider I_1 :

$$I_1 = \left\| \partial_t^2 u + \left(\tau^2 {f'}^2 + e^{2t} W \right) u + \Delta_\theta u \right\|_f^2.$$

Let $\rho > 0$ a small number to be chosen later. Since $|f''| \leq 1$ and $\tau \geq 1$, we have :

$$I_1 \geq \frac{\rho}{\tau} I_1', \tag{2.18}$$

where I_1' is defined by :

$$I_1' = \left\| \sqrt{|f''|} \left[\partial_t^2 u + \left(\tau^2 {f'}^2 + e^{2t} W \right) u + \Delta_\theta u \right] \right\|_f^2 \tag{2.19}$$

and one has

$$I_1' = K_1 + K_2 + K_3, \tag{2.20}$$

with

$$\begin{array}{rcl} K_1 & = & \left\| \sqrt{|f''|} \left(\partial_t^2 u + \Delta_\theta u \right) \right\|_f^2, \\ K_2 & = & \left\| \sqrt{|f''|} \left(\tau^2 {f'}^2 + e^{2t} W \right) u \right\|_f^2, \\ K_3 & = & 2 \left\langle \left(\partial_t^2 u + \Delta_\theta u \right) |f''|, \left(\tau^2 {f'}^2 + e^{2t} W \right) u \right\rangle_f. \end{array} \tag{2.21}$$

Integrating by parts gives :

$$
\begin{array}{rcl}
K_3 & = & 2\int f''\left(\tau^2 f'^2 + e^{2t}W\right)|\partial_t u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\
& + & 2\int \partial_t\left[f''\left(\tau^2 f'^2 + e^{2t}W\right)\right]\partial_t uu\sqrt{\gamma}f'^{-3}dtd\theta \\
& - & 6\int\left(f''^2 f'^{-1}\left(\tau^2 f'^2 + e^{2t}W\right)\right)\partial_t uu\sqrt{\gamma}f'^{-3}dtd\theta \\
& + & 2\int f''\left(\tau^2 f'^2 + e^{2t}W\right)\partial_t \ln\sqrt{\gamma}\partial_t uuf'^{-3}\sqrt{\gamma}dtd\theta \\
& + & 2\int f''\left(\tau^2 f'^2 + e^{2t}W\right)|D_\theta u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\
& + & 2\int f'' e^{2t}\partial_i W\cdot\gamma^{ij}\partial_j uuf'^{-3}\sqrt{\gamma}dtd\theta.
\end{array}
\tag{2.22}
$$

The condition $\tau \geq C_1\sqrt{\|W\|_{C^1}} + C_2$ implies,

$$|\partial_i W\gamma^{ij}\partial_j uu| \leq c\tau^2(|D_\theta u|^2 + |u|^2).$$

Now since $2\partial_t uu \leq u^2 + |\partial_t u|^2$, we can use conditions (2.1) and (2.5) to get

$$K_3 \geq -c\tau^2\int |f''|\left(|\partial_t u|^2 + |D_\theta u|^2 + |u|^2\right)f'^{-3}\sqrt{\gamma}dtd\theta \tag{2.23}$$

We also have

$$K_2 \geq c\tau^4\int |f''||u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \tag{2.24}$$

and since $K_1 \geq 0$,

$$
\begin{array}{rcl}
I_1 & \geq & -\rho c\tau\int |f''|\left(|\partial_t u|^2 + |D_\theta u|^2\right)f'^{-3}\sqrt{\gamma}dtd\theta \\
& + & C\tau^3\rho\int |f''||u|^2 f'^{-3}\sqrt{\gamma}dtd\theta.
\end{array}
\tag{2.25}
$$

Then using (2.17) and (2.25)

$$
\begin{array}{rcl}
I^2 & \geq & 4\tau^2\|f'\partial_t u\|_f^2 + 3\tau\int |f''||D_\theta u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\
& + & C\tau^3\rho\int |f''||u|^2 f'^{-3}\sqrt{\gamma}dtd\theta - c\tau^3\int e^t|u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \\
& - & \rho c\tau\int |f''|\left(|u|^2 + |\partial_t u|^2 + |D_\theta u|^2\right)f'^{-3}\sqrt{\gamma}dtd\theta. \\
& - & c\tau\int |f''||\partial_t u|^2 f'^{-3}\sqrt{\gamma}dtd\theta
\end{array}
\tag{2.26}
$$

Now one needs to check that every non-positive term in the right hand side of (2.26) can be absorbed in the first three terms.
First fix ρ small enough such that

$$\rho c\tau\int |f''|\cdot|D_\theta u|^2 f'^{-3}\sqrt{\gamma}dtd\theta \leq 2\tau\int |f''|\cdot|D_\theta u|^2 f'^{-3}\sqrt{\gamma}dtd\theta$$

where c is the constant appearing in (2.26). The other terms in the last integral of (2.26) can then be absorbed by comparing powers of τ (for C_2

large enough). Finally since conditions (2.1) imply that e^t is small compared to $|f''|$, we can absorb $-c\tau^3 e^t|u|^2$ in $C\tau^3\rho|f''||u|^2$.
Thus we obtain :

$$\begin{array}{rcl} I^2 & \geq & C\tau^2\int|\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta + C\tau\int|f''||D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ & + & C\tau^3\int|f''||u|^2{f'}^{-3}\sqrt{\gamma}dtd\theta \end{array} \tag{2.27}$$

As before, we can check that II can be absorbed in I for $|T_0|$ and τ large enough. Then we obtain

$$\|L_\tau u\|_f^2 \geq C\tau^3\|\sqrt{|f''|}u\|_f^2 + C\tau^2\|\partial_t u\|_f^2 + C\tau\|\sqrt{|f''|}D_\theta u\|_f^2. \tag{2.28}$$

Note that, since τ is large and $\sqrt{|f''|}\leq 1$, one has

$$\|L_\tau u\|_f^2 \geq C\tau^3\|\sqrt{|f''|}u\|_f^2 + c\tau\|\sqrt{|f''|}\partial_t u\|_f^2 + C\tau\|\sqrt{|f''|}D_\theta u\|_f^2, \tag{2.29}$$

and the constant c can be choosen arbitrary smaller than C. If we set $v = e^{-\tau\phi}u$, then we have

$$\begin{array}{rcl} \|e^{2t}e^{\tau\phi}(\Delta v + Wv)\|_f^2 & \geq & C\tau^3\|\sqrt{|f''|}e^{\tau\phi}v\|_f^2 - c\tau^3\|\sqrt{|f''|}f'e^{\tau\phi}v\|_f^2 \\ +\frac{c}{2}\tau\|\sqrt{|f''|}e^{\tau\phi}\partial_t v\|_f^2 & + & C\tau\|\sqrt{|f''|}e^{\tau\phi}D_\theta v\|_f^2 \end{array}.$$

Finally since f' is close to 1 one can absorb the negative term to obtain

$$\begin{array}{rcl} \|e^{2t}e^{\tau\phi}(\Delta v + Wv)\|_f^2 & \geq & C\tau^3\|\sqrt{|f''|}e^{\tau\phi}v\|_f^2 \\ +C\tau\|\sqrt{|f''|}e^{\tau\phi}\partial_t v\|_f^2 & + & C\tau\|\sqrt{|f''|}e^{\tau\phi}D_\theta v\|_f^2 \end{array}. \tag{2.30}$$

It remains to get back to the usual L^2 norm. First note that since f' is close to 1 (2.1), we can get the same estimate without the term $(f')^{-3}$ in the integrals. Recall that in polar coordinates (r,θ) the volume element is $r^{n-1}\sqrt{\gamma}drd\theta$, we can deduce from (2.27) by substitution that :

$$\begin{array}{rcl} \|r^2e^{\tau\phi}(\Delta v + Wv)r^{-\frac{n}{2}}\|^2 & \geq & C\tau^3\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}vr^{-\frac{n}{2}}\|^2 \\ & + & C\tau\|r^{1+\frac{\varepsilon}{2}}e^{\tau\phi}\nabla vr^{-\frac{n}{2}}\|^2. \end{array} \tag{2.31}$$

Finally one can get rid of the term $r^{-\frac{n}{2}}$ by replacing τ with $\tau+\frac{n}{2}$. Indeed from $e^{\tau\phi}r^{-\frac{n}{2}} = e^{(\tau+\frac{n}{2})\phi}e^{-\frac{n}{2}r^\varepsilon}$ one can check easily that, for r small enough

$$\frac{1}{2}e^{(\tau+\frac{n}{2})\phi} \leq e^{\tau\phi}r^{-\frac{n}{2}} \leq e^{(\tau+\frac{n}{2})\phi}.$$

This achieves the proof of the first part of theorem 2.1.

Now suppose that $\mathrm{supp}(u)\subset\{x\in M; r(x)\geq\delta>0\}$ and define $T_1 = \ln\delta$.

Cauchy-Schwarz inequality apply to

$$\int\partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = 2\int u\partial_t ue^{-t}\sqrt{\gamma}dtd\theta$$

gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \leq 2\left(\int (\partial_t u)^2\, e^{-t}\sqrt{\gamma}dtd\theta\right)^{\frac{1}{2}}\left(\int u^2 e^{-t}\sqrt{\gamma}dtd\theta\right)^{\frac{1}{2}}. \tag{2.32}$$

On the other hand, integrating by parts gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = \int u^2 e^{-t}\sqrt{\gamma}dtd\theta - \int u^2 e^{-t}\partial_t(\ln(\sqrt{\gamma}))\sqrt{\gamma}dtd\theta. \tag{2.33}$$

Now since $|\partial_t \ln\sqrt{\gamma}| \leq Ce^t$ for $|T_0|$ large enough we can deduce :

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \geq c\int u^2 e^{-t}\sqrt{\gamma}dtd\theta. \tag{2.34}$$

Combining (2.32) and (2.34) gives

$$\begin{aligned} c^2\int u^2 e^{-t}\sqrt{\gamma}dtd\theta &\leq 4\int (\partial_t u)^2\, e^{-t}\sqrt{\gamma}dtd\theta \\ &\leq 4e^{-T_1}\int (\partial_t u)^2\,\sqrt{\gamma}dtd\theta. \end{aligned}$$

Finally, droping all terms except $\tau^2 \int |\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta$ in (2.27) gives :

$$C' I^2 \geq \tau^2\delta^2 \|e^{-t}u\|_f^2.$$

Inequality (2.27) can then be replaced by :

$$\begin{array}{rcl} I^2 & \geq & C\tau^2 \int |\partial_t u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta + C\tau\int |f''|\cdot|D_\theta u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta \\ & + & C\tau^3\int |f''|\cdot|u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta + C\tau^2\delta^2\int |u|^2 {f'}^{-3}\sqrt{\gamma}dtd\theta. \end{array} \tag{2.35}$$

The rest of the proof follows in a similar way than the first part. □

Now we will establish a Carleman estimate for the operator $\Delta+\lambda$ acting on vector functions, which will be useful in the next section. For $U \in \mathcal{C}_0^\infty(B_{R_0}(x_0)\setminus\{x_0\},\mathbb{R}^m)$, applying (2.2) to each components U^i of U and summing gives :

Corollary 2.2. *There exist non-negative constants R_0, C, C_1, which depend only on M and ε, such that :*
$\forall x_0 \in M$, $\forall\, U \in \mathcal{C}_0^\infty(B_{R_0}(x_0)\setminus\{x_0\},\mathbb{R}^m)$, $\forall\, \tau \geq C_1\sqrt{\lambda}$,

$$\begin{aligned} C\left\|r^2 e^{-\tau\phi}\left(\Delta U+\lambda U\right)\right\| &\geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}}e^{-\tau\phi}U\right\| \\ &+ \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}}e^{-\tau\phi}\nabla U\right\| \end{aligned} \tag{2.36}$$

Moreover,

$$\text{If } \operatorname{supp}(U) \subset \{x \in M; r(x) \geq \delta > 0\},$$

then

$$\begin{aligned} C \left\| r^2 e^{-\tau\phi} (\Delta U + \lambda U) \right\| &\geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{-\tau\phi} U \right\| \\ + \tau\delta \left\| r^{-1} e^{-\tau\phi} U \right\| &+ \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{-\tau\phi} \nabla U \right\|. \end{aligned} \tag{2.37}$$

3 Doubling inequality

In this section we intend to prove a doubling property for gradient of eigenfunctions. First we establish a three sphere theorem :

Proposition 3.1 (Three spheres theorem)**.** *There exist non-negative constants R_0, c and $0 < \alpha < 1$ wich depend only on M such that, if u is a solution to* (1.1) *one has :*
$\forall R;\ 0 < R < 2R < R_0, \forall x_0 \in M,$

$$\|\nabla u\|_{B_R(x_0)} \leq e^{c\sqrt{\lambda}} \|\nabla u\|^{\alpha}_{B_{\frac{R}{2}}(x_0)} \|\nabla u\|^{1-\alpha}_{B_{2R}(x_0)} \tag{3.1}$$

Proof. Let x_0 a point in M and $(x_1, x_2, \cdots, x_n)$ local coordinates around x_0. Let u be a solution to (1.1) and define $V = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{x_n})$. Let $R_0 > 0$ as in theorem (2.2) and R such that $0 < R < 2R < R_0$. We still denote $r(x)$ the riemannian distance beetween x and x_0. We also denote by B_r the geodesic ball centered at x_0 of radius r. If v is a function defined in a neigborhood of x_0, we denote by $\|v\|_R$ the L^2 norm of v on B_R and by $\|v\|_{R_1,R_2}$ the L^2 norm of v on the set $A_{R_1,R_2} := \{x \in M;\ R_1 \leq r(x) \leq R_2\}$. Let $\psi \in \mathcal{C}_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a radial function with the following properties :

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or if $r(x) > \frac{5R}{3}$,
- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla\psi(x)| \leq \frac{C}{R}$, $|\nabla^2\psi(x)| \leq \frac{C}{R^2}$.

We recall that $\phi(x) = -\ln r(x) + r(x)^\varepsilon$.

First apply ∂_k to each side of (1.1) to get

$$\Delta\partial_k u - [\Delta, \partial_k]u = \partial_k u$$

where $[\Delta, \partial_k]$ is a second order operator with no zero order term and with coefficients depending only of M. The function $V = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{x_n})$ is therefore a solution of the system :

$$\Delta V + \lambda V - AV - B \cdot \nabla V = 0 \tag{3.2}$$

where A and B depend only on the metric g of M and its derivatives. Now we apply the Carleman estimate (2.37) to the function ψV with $f(t) = t - e^{\varepsilon t}$. We get :

$$\begin{aligned} C\left\|r^2 e^{\tau\phi}\left(\Delta(\psi V)+\lambda\psi V\right)\right\| &\geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}} e^{\tau\phi}\psi V\right\| \\ +\tau R\left\|r^{-1} e^{\tau\phi}\psi V\right\| &+ \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\nabla(\psi V)\right\|. \end{aligned}$$

Using that V is a solution of (3.2), we have :

$$\begin{aligned} C\left\|r^2 e^{\tau\phi}\left(\psi A V+\psi B\cdot\nabla V+2\nabla V\cdot\nabla\psi+\Delta\psi V\right)\right\| &\geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}} e^{\tau\phi}\psi V\right\| \\ +\tau R\left\|r^{-1} e^{\tau\phi}\psi V\right\|+\tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\nabla(\psi V)\right\| & \end{aligned}$$

Now from triangular inequality we get

$$\begin{aligned} C\left\|r^2 e^{\tau\phi}\left(\Delta\psi V+2\nabla V\cdot\nabla\psi\right)\right\| \geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}} e^{\tau\phi}\psi V\right\| - C\left\|r^2 e^{\tau\phi}\psi A V\right\| \\ +\tau R\left\|r^{-1} e^{\tau\phi}\psi V\right\|+\tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{-\tau\phi}\nabla(\psi V)\right\| - C\left\|r^2 e^{\tau\phi}\psi B\cdot\nabla V\right\| \end{aligned}$$

and

$$\begin{aligned} \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\nabla(\psi V)\right\| &\geq \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\psi\nabla V\right\| - \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\nabla\psi V\right\| \\ &\geq \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\psi\nabla V\right\| - \tau^{\frac{1}{2}}\left\|r^{\frac{\varepsilon}{2}} e^{\tau\phi} V\right\| \end{aligned}$$

Then for τ great enough and for sufficient small R_0 ,

$$\begin{aligned} C\left\|r^2 e^{\tau\phi}\left(\Delta\psi V+2\nabla V\cdot\nabla\psi\right)\right\| &\geq \tau^{\frac{3}{2}}\left\|r^{\frac{\varepsilon}{2}} e^{-\tau\phi}\psi V\right\| \\ +\tau R\left\|r^{-1} e^{-\tau\phi}\psi V\right\| &+ \tau^{\frac{1}{2}}\left\|r^{1+\frac{\varepsilon}{2}} e^{-\tau\phi}\psi\nabla V\right\|. \end{aligned} \tag{3.3}$$

In particular we have :

$$C\left\|r^2 e^{\tau\phi}\left(\Delta\psi V+2\nabla V\cdot\nabla\psi\right)\right\| \geq \tau\left\|e^{\tau\phi}\psi V\right\|$$

Assume that $\tau \geq 1$, and use properties of ψ to get :

$$\begin{aligned} \|e^{\tau\phi}V\|_{\frac{R}{3},\frac{3R}{2}} &\leq C\left(\|e^{\tau\phi}V\|_{\frac{R}{4},\frac{R}{3}}+\|e^{\tau\phi}V\|_{\frac{3R}{2},\frac{5R}{3}}\right) \\ &+ C\left(R\|e^{\tau\phi}\nabla V\|_{\frac{R}{4},\frac{R}{3}}+R\|e^{\tau\phi}\nabla V\|_{\frac{3R}{2},\frac{5R}{3}}\right). \end{aligned} \tag{3.4}$$

Furthermore as ϕ is radial and decreasing,

$$\begin{aligned} \|e^{\tau\phi}V\|_{\frac{R}{3},\frac{3R}{2}} &\leq C\left(e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{4},\frac{R}{3}}+e^{\tau\phi(\frac{3R}{2})}\|V\|_{\frac{3R}{2},\frac{5R}{3}}\right) \\ &+ C\left(Re^{\tau\phi(\frac{R}{4})}\|\nabla V\|_{\frac{R}{4},\frac{R}{3}}+Re^{\tau\phi(\frac{3R}{2})}\|\nabla V\|_{\frac{3R}{2},\frac{5R}{3}}\right). \end{aligned}$$

Now we recall the following elliptic estimates : since V satisfies (3.2) then hard to see that :

$$\|\nabla V\|_{(1-a)r} \leq C\left(\frac{1}{(1-a)R}+\sqrt{\lambda}\right)\|V\|_{B_R}, \quad \text{for } 0<a<1 \tag{3.5}$$

As $\|e^{\tau\phi}\nabla V\|_{\frac{R}{4},\frac{R}{3}}$ is bounded by $\|e^{\tau\phi}\nabla V\|_{\frac{R}{3}}$, using the formula (3.5) gives :

$$e^{\tau\phi(\frac{R}{4})}\|\nabla V\|_{\frac{R}{4},\frac{R}{3}} \leq C\left(\frac{1}{R}+\sqrt{\lambda}\right)e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{2}},$$

Simiraly, we have also,

$$e^{\tau\phi(\frac{3R}{2})}\|\nabla V\|_{\frac{3R}{2},\frac{5R}{3}} \leq C\left(\frac{1}{R}+\sqrt{\lambda}\right)e^{\tau\phi(\frac{3R}{2})}\|V\|_{2R}.$$

Using properties of ϕ :

$$\|e^{\tau\phi}V\|_{\frac{R}{3},\frac{3R}{2}} \geq \|e^{\tau\phi}V\|_{\frac{R}{3},R} \geq e^{\tau\phi(R)}\|V\|_{\frac{R}{3},R}.$$

Using (3.4) one has :

$$\|V\|_{\frac{R}{3},R} \leq C\sqrt{\lambda}\left(e^{\tau(\phi(\frac{R}{4})-\phi(R))}\|V\|_{\frac{R}{2}}+e^{\tau(\phi(\frac{3R}{2})-\phi(R))}\|V\|_{2R}\right)$$

Let $A_R=\phi(\frac{R}{4})-\phi(R)$ and $B_R=-(\phi(\frac{3R}{2})-\phi(R))$. Because of the properties of ϕ, we have $0<C_1\leq A_R\leq C_2$ and $0<C_1\leq B_R\leq C_2$ where C_1 and C_2 don't depend on R. We may assume that $C\sqrt{\lambda}\geq 2$. We can add $\|V\|_{\frac{R}{3}}$ to each member and bound it in the right hand side by $C\sqrt{\lambda}e^{\tau A}\|V\|_{\frac{R}{2}}$. Then replacing C by $2C$ gives :

$$\|V\|_R \leq C\sqrt{\lambda}e^{\tau A}\|V\|_{\frac{R}{2}}+\|V\|_{\frac{R}{3}}+C_\lambda e^{-\tau B}\|V\|_{2R} \tag{3.6}$$

$$\|V\|_R \leq C\sqrt{\lambda}\left(e^{\tau A}\|V\|_{\frac{R}{2}}+e^{-\tau B}\|V\|_{2R}\right). \tag{3.7}$$

Now we want to find τ such that

$$C\sqrt{\lambda}e^{-\tau B}\|V\|_{2R} \leq \frac{1}{2}\|V\|_R$$

wich is true for $\tau \geq -\frac{1}{B}\ln\left(\frac{1}{2C\sqrt{\lambda}}\frac{\|V\|_R}{\|V\|_{2R}}\right)$. Since τ must satisfy

$$\tau \geq C_1\sqrt{\lambda},$$

we choose

$$\tau = -\frac{1}{B}\ln\left(\frac{1}{2C\sqrt{\lambda}}\frac{\|V\|_R}{\|V\|_{2R}}\right)+C_1\sqrt{\lambda}. \tag{3.8}$$

Inequality (3.6) becomes

$$\|V\|_R \leq C\sqrt{\lambda}e^{C_1\sqrt{\lambda}}e^{\frac{-A}{B}\ln\left(\frac{1}{2C_\lambda}\frac{\|V\|_R}{\|V\|_{2R}}\right)}\|V\|_{\frac{R}{2}},$$

$$\|V\|_R \leq e^{(C_1\sqrt{\lambda})\frac{B}{A+B}}\|V\|_{2R}^{\frac{A}{A+B}}\|V\|_{\frac{R}{2}}^{\frac{B}{B+A}}.$$

Finally define $\alpha = \frac{A}{A+B}$ and replace C_i by $C_i\frac{B}{A+B}$ to have

$$\|V\|_R \leq e^{C_5\sqrt{\lambda}}\|V\|_{2R}^{\alpha}\|V\|_{\frac{R}{2}}^{1-\alpha}.$$

□

From now on we assume that M is compact. Thus we can derive from three sphere theorem above uniform doubling estimates on solutions.

Theorem 3.2 (doubling estimates)**.** *There exist two non-negative constants R_0, C_1 depending only on M such that : if u is a solution to* (1.1) *on M then $\forall x_0 \in M, \forall r > 0$,*

$$\|\nabla u\|_{B_{2r}(x_0)} \leq e^{C_1\sqrt{\lambda}}\|\nabla u\|_{B_r(x_0)}. \tag{3.9}$$

Remark 3.3. *Using standard elliptic theory to bound the L^∞ norm of $|V|$ by a multiple of its L^2 norm gives for $\delta > 0$:*

$$\|V\|_{L^\infty(B_\delta(x_0))} \geq (C_1\lambda + C_2)^{\frac{n}{2}}\delta^{-n/2}\|u\|_{2\delta}$$

Then one can see that the doubling estimate is still true with the L^∞ norm

$$\|V\|_{L^\infty(B_{2r}(x_0))} \leq e^{C\sqrt{\lambda}}\|V\|_{L^\infty(B_r(x_0))} \tag{3.10}$$

To proove the theorem 3.2 we need the following

Proposition 3.4. $\forall R > 0,\ \exists\ C_R > 0,\ \forall x_0 \in M$:

$$\|\nabla u\|_{B_R(x_0)} \geq e^{-C_R\sqrt{\lambda}}\|\nabla u\|_{L^2(M)}.$$

Proof. Let $R > 0$ and assume without loss of generality that $R < R_0$ whith R_0 such that three spheres theorem (theorem 3.1) is valid. Up to multiplication by a constant, we can assume that $\|\nabla u\|_{L^2(M)} = 1$. We denote by $\bar{x}$ a point in M such that $\|\nabla u\|_{B_R(\bar{x})} = \sup_{x\in M}\|\nabla u\|_{B_R(x)}$. This implies that one has $\|\nabla u\|_{B_{R(\bar{x})}} \geq D_R$, where D_R depend only on M and R. One has from proposition (3.1) at an arbitrary point x of M :

$$\|\nabla u\|_{B_{R/2}(x)} \geq e^{-c\sqrt{\lambda}}\|\nabla u\|_{B_R(x)}^{\frac{1}{\alpha}} \tag{3.11}$$

Let γ be a geodesic curve beetween x and $\bar{x}$ and define $x_0 = x,\ x_1, \cdots, x_m = \bar{x}$ such that $x_i \in \gamma$ and $B_{\frac{R}{2}}(x_{i+1}) \subset B_R(x_i),\ \forall i = 1, \cdots, m$. The constant m depends only on $\mathrm{diam}(M)$ and R. Then the properties of $(x_i)_{1\leq i \leq m}$ and inequality (3.11) give for all i, $1 \leq i \leq m$:

$$\|\nabla u\|_{B_{R/2}(x_i)} \geq e^{-c_i\sqrt{\lambda}} \|\nabla u\|^{\frac{1}{\alpha}}_{B_{R/2}(x_{i+1})}. \tag{3.12}$$

The result follows by induction and the fact that $\|\nabla u\|_{B_R(\bar{x})} \geq D_R$.

□

Corollary 3.5. *For all $R > 0$, there exists a positive constant C_R depending only on M and R such that at any point x_0 in M one has*

$$\|\nabla u\|_{\frac{R}{4},\frac{R}{8}} \geq e^{-C_R\sqrt{\lambda}} \|\nabla u\|_{L^2(M)}$$

Proof. Let $R < R_0$ where R_0 is such that the three spheres theorem is valid, note that $R_0 \leq \mathrm{diam}(M)$. Recall that we defined locally near a point x_0 : $A_{r_1,r_2} := \{x \in M; r_1 \leq d(x, x_0) \leq r_2)\}$. As M is geodesically complete, there exists a point x_1 in $A_{\frac{R}{8},\frac{R}{4}}$ such that $B_{x_1}(\frac{R}{16}) \subset A_{\frac{R}{8},\frac{R}{4}}$. From proposition 3.4 one has $\|\nabla u\|_{B_{\frac{R}{16}}(x_1)} \geq e^{-C_R\sqrt{\lambda}}\|\nabla u\|_{L^2(M)}$ wich gives the result. □

Proof of theorem 3.2. We proceed like in the proof of three spheres theorem except for the fact that now we want the first ball to become arbitrary small in front of the others. Let $R = \frac{R_0}{4}$ where R_0 is such that the three spheres theorems is valid, let δ such that $0 < \delta < 2\delta < 3\delta < \frac{R}{8} < \frac{R}{2} < R$, and define a smooth radial function ψ, with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or if $r(x) > R$,
- $\psi(x) = 1$ if $\frac{5\delta}{4} < r(x) < \frac{R}{2}$,
- $|\nabla\psi(x)| \leq \frac{C}{\delta}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$ and $|\nabla\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$,
- $|\nabla^2\psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$ and $|\nabla^2\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$.

Keeping appropriates terms in (3.3) gives :

$$\begin{aligned}
\|r^{\frac{\varepsilon}{2}} e^{\tau\phi}\psi V\| + \tau\delta\|r^{-1}e^{\tau\phi}\psi V\| &\leq C\left(\|r^2 e^{\tau\phi}\nabla V\cdot\nabla\psi\| + \|r^2 e^{\tau\phi}\Delta\psi V\|\right) \\
&\leq \frac{C}{\delta}\|r^2 e^{\tau\phi}\nabla V\|_{\delta,\frac{5\delta}{4}} + C\|e^{\tau\phi}\nabla V\|_{\frac{R}{2},R} \\
&+ \frac{C}{\delta^2}\|r^2 e^{\tau\phi} V\|_{\delta,\frac{5\delta}{4}} + C\|e^{\tau\phi} V\|_{\frac{R}{2},R}
\end{aligned}$$

Using properties of ψ we have,

$$\begin{aligned}
& \|r^{\frac{\varepsilon}{2}}e^{\tau\phi}V\|_{\frac{5\delta}{4},3\delta} + \|r^{\frac{\varepsilon}{2}}e^{\tau\phi}V\|_{\frac{R}{8},\frac{R}{4}} \\
+ \quad & \tau\delta\|r^{-1}e^{\tau\phi}V\|_{\frac{5\delta}{4},3\delta} + \tau\delta\|r^{-1}e^{\tau\phi}V\|_{\frac{R}{8},\frac{R}{4}} \\
\leq \quad & \frac{C}{\delta}\|r^2e^{\tau\phi}\nabla V\|_{\delta,\frac{5\delta}{4}} + C\|e^{\tau\phi}\nabla V\|_{\frac{R}{2},R} \\
+ \quad & \frac{C}{\delta^2}\|r^2e^{\tau\phi}V\|_{\delta,\frac{5\delta}{4}} + C\|e^{\tau\phi}V\|_{\frac{R}{2},R}.
\end{aligned}$$

Now drop the first and last terms of the left hand side gives :

$$\begin{aligned}
\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}V\|_{\frac{R}{8},\frac{R}{4}} + \|e^{\tau\phi}V\|_{\frac{5\delta}{4},3\delta} \ \leq \ & C\left(\delta\|e^{\tau\phi}\nabla V\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}\nabla V\|_{\frac{R}{2},R}\right) \\
+ \ & C\left(\|e^{\tau\phi}V\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}V\|_{\frac{R}{2},R}\right)
\end{aligned}$$

Now using (3.5) and properties of ϕ,

$$\begin{aligned}
\|e^{\tau\phi}V\|_{\frac{R}{8},\frac{R}{4}} + \|e^{\tau\phi}V\|_{\frac{5\delta}{4},3\delta} \ \leq \ & C\sqrt{\lambda}\left(e^{\tau\phi(\delta)}\|\ V\|_{\frac{2\delta}{3},\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{R}{5},\frac{5R}{3}}\right) \\
+ \ & C\sqrt{\lambda}\left(e^{\tau\phi(\delta)}\|V\|_{\delta,\frac{5\delta}{4}} + e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{R}{2},R}\right)
\end{aligned}$$

$$\|e^{\tau\phi}V\|_{\frac{R}{8},\frac{R}{4}} + \|e^{\tau\phi}V\|_{\frac{5\delta}{4},3\delta} \leq C\sqrt{\lambda}\left(e^{\tau\phi(\delta)}\|V\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{5R}{3}}\right)$$

$$e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{8},\frac{R}{4}} + e^{\tau\phi(3\delta)}\|V\|_{\frac{5\delta}{4},3\delta} \leq C\sqrt{\lambda}\left(e^{\tau\phi(\delta)}\|V\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{5R}{3}}\right)$$

Adding $e^{\tau\phi(3\delta)}\|V\|_{\frac{5\delta}{4}}$ to each side

$$e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{8},\frac{R}{4}} + e^{\tau\phi(3\delta)}\|V\|_{3\delta} \leq C\sqrt{\lambda}\left(e^{\tau\phi(\delta)}\|V\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{5R}{3}}\right)$$

Now we want to choose τ such that

$$C\sqrt{\lambda}e^{\tau\phi(\frac{R}{5})}\|V\|_{\frac{5R}{3}} \leq \frac{1}{2}e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{8},\frac{R}{4}}$$

For the same reasons than before we choose

$$\tau = \frac{1}{\phi(\frac{R}{5}) - \phi(\frac{R}{4})}\ln\left(\frac{1}{2C\sqrt{\lambda}}\frac{\|u\|_{\frac{R}{8},\frac{R}{4}}}{\|u\|_{\frac{5R}{3}}}\right) + C_1\sqrt{\lambda}$$

Define $A = \left(\phi(\frac{R}{5}) - \phi(\frac{R}{4})\right)^{-1}$; like before one can assume that A is non-positive and independent of R. So,

$$e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{8},\frac{R}{4}} + e^{\tau\phi(3\delta)}\|V\|_{3\delta} \leq C\sqrt{\lambda}e^{\tau\phi(\delta)}\|V\|_{\frac{5\delta}{2}}$$

One can then ignore the first term of the right hand side to get :

$$e^{\tau\phi(3\delta)}\|V\|_{3\delta} \leq C\sqrt{\lambda}\; e^{A\ln\left(\frac{1}{2C\sqrt{\lambda}}\frac{\|V\|_{\frac{R}{8},\frac{R}{4}}}{\|V\|_{\frac{5R}{3}}}\right)+C_1\sqrt{\lambda}}\|V\|_{\frac{3\delta}{2}}$$

$$\|V\|_{3\delta} \leq e^{C\sqrt{\lambda}}\left(\frac{\|V\|_{\frac{R}{8},\frac{R}{4}}}{\|V\|_{\frac{5R}{3}}}\right)^{A}\|V\|_{\frac{3\delta}{2}}$$

Finally from corollary 3.5, define $r=\frac{3\delta}{2}$ to have :

$$\|V\|_{2r} \leq e^{C\sqrt{\lambda}}\|V\|_{r}$$

Thus, the theorem is proved for all $r \leq \frac{R_0}{16}$. Using proposition 3.4 we have for $r \geq \frac{R_0}{16}$:

$$\|\nabla u\|_{B_{x_0}(r)} \geq \|\nabla u\|_{B_{x_0}(\frac{R_0}{16})} \geq e^{-C_0\sqrt{\lambda}}\|\nabla u\|_{L^2(M)} \geq e^{-C_1\sqrt{\lambda}}\|\nabla u\|_{B_{x_0}(2r)}$$

$\square$

4 Critical set on analytic manifold

From here we will follow the method of Donnelly and Fefferman [6] to establish upper bound for the $(n-1)$-dimensionnal measure of critical set of eigenfunctions. So we also suppose that M is analytic. Recall that $\mathcal{N}_u = \{x \in M : u(x)=0\}$ and $\mathcal{C}_u = \{x \in M : \nabla u(x) = 0\}$. Define $B_{\mathbb{C}}(r)$ the complex ball :

$$B_{\mathbb{C}}(r) = \{z \in \mathbb{C}^n : |z| < r\}$$

and $B(r)$ the standard ball in $\mathbb{R}^n$ centred at 0 of radius r. The main point to deduce from our doubling inequality an estimate on the Hausdorff measure of the critical set is the following result of Donnelly and Fefferman :

Theorem 4.1 ([6] p. 180)**.** *Let F be an holomorphic function on $B_{\mathbb{C}}(1)$ and suppose there exists $\alpha > 1$ such that*

$$\max_{B_{\mathbb{C}}(1)}|F| \leq e^{\alpha}\max_{B(\frac{1}{2})}|F|,$$

then

$$\mathcal{H}^{n-1}\left(\mathcal{N}_F \cap B\left(\frac{1}{4}\right)\right) \leq C\alpha.$$

where $\mathcal{N}_F$ is the zero set of F in $\mathbb{R}^n$ and C a constant depending only on the dimension.

Let u be a solution to (1.1). Fix x_0 in M and consider $(x_1, \cdots, x_n)$ a chart around x_0. We assume that the chart contains the euclidean ball B_2. We define

$$F(x) = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2,$$

The nodal set of F is the critical set of u. One has :

Proposition 4.2. *The function F can be extended to an analytic function on $B_{\mathbb{C}}(1)$ and :*

$$\|F\|_{L^\infty(B_{\mathbb{C}}(1))} \leq e^{C\sqrt{\lambda}} \|F\|_{L^\infty(B(\frac{1}{2}))}$$

where C is a constant depending only on M.

Lemma 4.3. *Let u be an eigenfunction of the laplace operator on $B(1)$, for all multi-index β, with $|\beta| \geq 1$ one has :*

$$|D^\beta u(0)| \leq \beta! C^{|\beta|} \sqrt{\lambda}^{|\beta|} \|\nabla u\|_{L^\infty\left(B(\frac{C_1}{\sqrt{\lambda}})\right)} \tag{4.1}$$

where C_1 is a constant small enough.

proof of lemma 4.3. Like in [6], this result can be obtained by rescaling the equation and using the hypoellipticity proof ([9], p.178) for an elliptic operator whose coefficients have uniform bounded derivatives.
Indeed note first that we may assume $\|\nabla u\|_{L^\infty(M)} = 1$. Now writing in our local chart $\Delta = \sum_{1 \leq |\alpha| \leq 2} a_\alpha D^\alpha$ and consider the function $u_\lambda(x) = u(\frac{C_1}{\sqrt{\lambda}} x)$, where C_1 will be fix below. One can see that u_λ is a solution to the elliptic equation

$$P_\lambda u_\lambda = u_\lambda$$

with $P_\lambda = \sum_{1 \leq |\alpha| \leq 2} b_\alpha D^\alpha$ and

$$b_\alpha(x) = \frac{\lambda^{-1+\frac{|\alpha|}{2}}}{C_1^{|\alpha|}} a_\alpha \left(\frac{C_1 x}{\sqrt{\lambda}} \right).$$

A short computation of $D^\beta b_\alpha$, gives for C_1 small enough and any multi-index β:

$$\sup_{B_1} |D^\beta b_\alpha(x)| \leq C_2 |\beta|!, \quad \forall 1 \leq |\alpha| \leq 2$$

where C_2 is a constant depending only on M. Then one can use the hypoellipticity proof ([9]) with simple modifications to get for any multi-index β with $|\beta| > 1$:

$$|D^\beta u_\lambda(0)| \leq A^{|\beta|} \beta!.$$

□

Proof of proposition 4.2. Expanding $V = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{x_n})$ in its Taylor series gives

$$V(z) = \sum_{|\alpha| \geq 0} \frac{z^\alpha}{\alpha!} D^\alpha V(0),$$

where for $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $\mathbb{N}^n$ and $z = (z_1, \cdots, z_n)$ in $\mathbb{C}^n$ we have set $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$. Now using (4.1) and summing a geometric series gives for a constant ρ small enough

$$\sup_{B_{\mathbb{C}}(0, \frac{\rho}{\sqrt{\lambda}})} |V(z)| \leq C \sup_{B(0, \frac{C_1}{\sqrt{\lambda}})} |V(x)|. \tag{4.2}$$

Then by translating, in the complex ball $B_{\mathbb{C}}(1)$, the equation and iterating the estimate (4.2) a multiple of $\sqrt{\lambda}$ times one has

$$\forall z \in B_{\mathbb{C}}(1), \ |V(z)| \leq C^{\sqrt{\lambda}} \sup_{B(2)} |V(x)|$$

This implies

$$\sup_{B_{\mathbb{C}}(1)} |F(z)| \leq e^{C\sqrt{\lambda}} \sup_{B(2)} |F(x)| \tag{4.3}$$

which gives proposition 4.2 by using doubling inequality (3.9). □

proof of theorem 1.2. Let u be a solution to (1.1), let $r_0 > 0$ a fixed number not larger than the injectivity radius of M and p a arbitrary point in M. Let consider a normal chart around p. By proposition 4.2 one has that $F = \sum_{i=1..n} \left| \frac{\partial u}{\partial x_i} \right|^2$ satisfy the hypothesis of theorem 4.1. Then since the nodal set of F is the critical set of u one has

$$\mathcal{H}^{n-1} \left(\mathcal{C}_u \cap B(p, r_0) \right) \leq C\sqrt{\lambda} \tag{4.4}$$

where C depends only on r_0 and M.
The Theorem 1.2 follows by a covering argument since M is compact. □

Remark 4.4. *Since doubling estimates imply vanishing order estimates it follows from lemma 3 of [3] that the local estimate* (4.4) *is still true on smooth manifold, but without any control on the radius* r_0.

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